

Use the binomial series to write the Maclaurin series for $f(x) = \sin^{-1} x$ in sigma notation.

SCORE: ____ / 25 PTS

Simplify all coefficients as shown in lecture.

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} = (1+(-x^2))^{-\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n}$$

$$\sin^{-1} x = C + \int \frac{1}{\sqrt{1-x^2}} dx$$

$$= C + \int \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n} dx = C + \sum_{n=0}^{\infty} \int (-1)^n \binom{-\frac{1}{2}}{n} x^{2n} dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1}$$

$$= C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n! (2n+1)} x^{2n+1}$$

For $n > 0$,

$$\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{1}{2}-n+1)}{n!}$$
$$= \frac{(-1)^n (1 \cdot 3 \cdot 5 \cdots (2n-1))}{2^n n!}$$

$$\binom{-\frac{1}{2}}{0} = 1$$

$$\sin^{-1} 0 = C$$

$$0 = C$$

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n! (2n+1)} x^{2n+1}$$

Let $f(x) = \frac{x^5}{7+x}$.

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- [a] Find a power series for $f(x)$. Write your answer in sigma notation.

NOTE: You do **NOT** need to use differentiation **NOR** the binomial series.

$$\begin{aligned}\frac{x^5}{7+x} &= \frac{x^5}{7} \left(\frac{1}{1+\frac{x}{7}} \right) = \frac{x^5}{7} \sum_{n=0}^{\infty} \left(-\frac{x}{7} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+5}}{7^{n+1}} \\ &= \sum_{n=5}^{\infty} \frac{(-1)^{n+1} x^n}{7^{n-4}}\end{aligned}$$

- [b] Find the interval of convergence for the power series for $f(x)$.

NOTE: You do **NOT** need to find a limit.

$$\begin{aligned}\left| -\frac{x}{7} \right| < 1 &\Rightarrow |x| < 7 \\ -7 &< x < 7\end{aligned}$$

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n-3)(x+7)^n}{n^2+4}$.

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$$\lim_{n \rightarrow \infty} \left| \frac{(2(n+1)-3)(x+7)^{n+1}}{(n+1)^2+4} \cdot \frac{n^2+4}{(2n-3)(x+7)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2n-1}{2n-3} \cdot \frac{n^2+4}{n^2+2n+5} (x+7) \right|$$

$$= |x+7| < 1$$

$$\text{IF } -1 < x+7 < 1$$

$$-8 < x < -6$$

$$x = -6: \sum_{n=0}^{\infty} \frac{2n-3}{n^2+4} \text{ COMPARE TO } \sum_{n=0}^{\infty} \frac{2n}{n^2} = \sum_{n=0}^{\infty} \frac{2}{n}$$

$$\frac{2n-3}{n^2+4} > 0 \text{ IF } n > \frac{3}{2} \text{ AND } \frac{2}{n} > 0 \text{ IF } n > 0$$

$$\lim_{n \rightarrow \infty} \frac{\frac{2n-3}{n^2+4}}{\frac{2}{n}} = \lim_{n \rightarrow \infty} \frac{2n^2-3n}{2n^2+8} = 1 \neq 0$$

$$\sum_{n=0}^{\infty} \frac{2}{n} \text{ DIVERGES (} p=1 \text{)}$$

$$\text{SO } \sum_{n=0}^{\infty} \frac{2n-3}{n^2+4} \text{ DIVERGES BY LIMIT COMPARISON TEST}$$

$$x = -8: \sum_{n=0}^{\infty} \frac{(-1)^n(2n-3)}{n^2+4}$$

$$\frac{2n-3}{n^2+4} > 0 \text{ IF } n > \frac{3}{2}$$

$$\lim_{n \rightarrow \infty} \frac{2n-3}{n^2+4} = 0$$

$$\frac{d}{dx} \frac{2x-3}{x^2+4} = \frac{2(x^2+4) - 2x(2x-3)}{(x^2+4)^2} = \frac{-2x^2+6x+8}{(x^2+4)^2} = \frac{-2(x-4)(x+1)}{(x^2+4)^2} < 0$$

$$\text{IF } x > 4$$

$$\text{SO } \left\{ \frac{2n-3}{n^2+4} \right\} \text{ IS DECREASING}$$

$$\text{SO } \sum_{n=0}^{\infty} \frac{(-1)^n(2n-3)}{n^2+4} \text{ CONVERGES BY ALTERNATING SERIES TEST}$$

$$\text{INTERVAL} = [-8, -6) \text{ OR } -8 \leq x < -6$$

Determine if the series $\sum_{n=2}^{\infty} \frac{(\ln n)(\sin n)}{n^2}$ converges.

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$$0 \leq \left| \frac{(\ln n)(\sin n)}{n^2} \right| \leq \frac{\ln n}{n^2}$$

$f(x) = \frac{\ln x}{x^2}$ IS POSITIVE + CONTINUOUS ON $[2, \infty)$

$$f'(x) = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3} < 0 \text{ IF } x > \sqrt{e}$$

SO f IS DECREASING ON $[2, \infty)$

$$\begin{aligned} \int_2^{\infty} \frac{\ln x}{x^2} dx &= \lim_{N \rightarrow \infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right) \Big|_2^N \\ &= \lim_{N \rightarrow \infty} \left(-\frac{\ln N}{N} - \frac{1}{N} + \frac{\ln 2}{2} - \frac{1}{2} \right) \\ &= 0 - 0 + \frac{\ln 2}{2} - \frac{1}{2} \end{aligned}$$

CONVERGES

$$\text{SO } \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \text{ CONVERGES BY INTEGRAL TEST} = \lim_{N \rightarrow \infty} \frac{-\frac{1}{2}}{1} = 0$$

SO $\sum_{n=2}^{\infty} \left| \frac{(\ln n)(\sin n)}{n^2} \right|$ CONVERGES BY COMPARISON TEST

SO $\sum_{n=2}^{\infty} \frac{(\ln n)(\sin n)}{n^2}$ CONVERGES BY ABSOLUTE CONVERGENCE TEST

$$\begin{array}{rcl} \frac{u}{v} & \frac{du}{v^2} & \\ \frac{\ln x}{x^{-1}} & \frac{x^{-2}}{-x^{-1}} & \\ \frac{1}{0} & \frac{-x^{-2}}{x^{-1}} & \end{array}$$

$$\lim_{N \rightarrow \infty} \frac{-\ln N}{N} = \frac{-\infty}{\infty}$$

Find the sum of the series $\frac{\pi}{5} + \frac{\pi^2}{2 \times 25} + \frac{\pi^3}{3 \times 125} + \frac{\pi^4}{4 \times 625} + \dots$.

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$$= \left[\left(-\frac{\pi}{5}\right) - \frac{\left(-\frac{\pi}{5}\right)^2}{2} + \frac{\left(-\frac{\pi}{5}\right)^3}{3} - \frac{\left(-\frac{\pi}{5}\right)^4}{4} + \dots \right]$$

$$= -\ln\left(1 + \frac{\pi}{5}\right)$$

$$= -\ln\left(1 - \frac{\pi}{5}\right)$$

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!(x-1)^n}{3^{2n}(n!)^2}$.

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$$\lim_{n \rightarrow \infty} \left| \frac{(2(n+1))!(x-1)^{n+1}}{3^{2(n+1)}((n+1)!)^2} \cdot \frac{3^{2n}(n!)^2}{(2n)!(x-1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)(x-1)}{3^2 (n+1)^2} \right|$$

$$= |x-1| \lim_{n \rightarrow \infty} \frac{1}{9} \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \frac{4}{9} |x-1| < 1$$

$$\text{IF } |x-1| < \frac{9}{4}$$

$$\text{RADIUS} = \frac{9}{4}$$

Find the first 4 non-zero terms of the Taylor series for $f(x) = \sec x$ about $x = \frac{\pi}{4}$.

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$$\frac{n}{0} \quad \frac{f^{(n)}(x)}{\sec x}$$

$$1 \quad \sec x \tan x$$

$$2 \quad \sec x \tan^2 x + \sec^3 x$$

$$3 \quad \sec x \tan^3 x + 2 \sec^3 x \tan x + 3 \sec^3 x \tan x \\ = \sec x \tan^3 x + 5 \sec^3 x \tan x$$

$$\frac{f^{(n)}\left(\frac{\pi}{4}\right)}{\sqrt{2}}$$

$$\sqrt{2}$$

$$\sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

$$\sqrt{2} + 5 \cdot 2\sqrt{2} = 11\sqrt{2}$$

$$\sec \frac{\pi}{4} = \sqrt{2}$$

$$\tan \frac{\pi}{4} = 1$$

$$\sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3$$